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Ordinal sums and idempotents of copulas

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Abstract. We prove that the ordinal sum of n-copulas is always an n-copula and show that every copula may be represented as an ordinal sum, once the set of its idempotents is known. In particular, it will be shown that every copula can be expressed as the ordinal sum of copulas having only trivial idempotents. As a by-product, we also characterize all associative copulas whose n-ary forms are n-copulas for all n.

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1. Introduction

The concept of *ordinal sum* was introduced in the algebraic framework of posets [2] and of semigroups [4,3]. It applies equally well to copulas. In the theory of triangular norms (briefly, t–norms) (see Refs. [13,8]) ordinal sums are used in order to provide both a general method of construction and a means of representing continuous t-norms. In analogy with the theory of t-norms one can show that both points of view are also valid for copulas.

The paper is organized as follows. In the next section we introduce the notion of ordinal sum for n-copulas ($n \geq 2$) and show that any such ordinal sum is again an n-copula. Then, in Sect. 3, the idempotents of an n-copula are used in order to represent any n-copula C as the ordinal sum of n-copulas having only trivial idempotents. Section 4 is devoted to the investigation of associative n-copulas. In the appendix we provide a direct proof for the case $n = 2$ of the fact that the ordinal sums of copulas is a copula.

2. Ordinal sums of copulas

We need to recall a few concepts. An n-*box* in $[0,1]^n$ is a cartesian product

$$[\mathbf{a}, \mathbf{b}] = \prod_{j=1}^{n} [a_j, b_j] ,$$

where, for every index $j \in \{1, 2, \ldots, n\}$, $0 \leq a_j \leq b_j \leq 1$.

For a function $C : [0,1]^n \to [0,1]$, the C-volume V_C of the box $[\mathbf{a},\mathbf{b}]$ is defined via

$$V_C([\mathbf{a},\mathbf{b}]) := \sum_{\mathbf{v}} \operatorname{sign}(\mathbf{v})\, C(\mathbf{v}) \tag{2.1}$$

where the sum is taken over all the 2^n vertices $\mathbf{v}$ of the box $[\mathbf{a},\mathbf{b}]$; here

$$\operatorname{sign}(\mathbf{v}) = \begin{cases} 1, & \text{if } v_j = a_j \text{ for an even number of indices,} \\ -1, & \text{if } v_j = a_j \text{ for an odd number of indices.} \end{cases}$$

A function $C : [0,1]^n \to [0,1]$ is said to be an n-copula if, and only if, it satisfies the following conditions:

(C1) $C_n(1,1,\dots,1,x_j,1,\dots,1) = x_j$, $j = 1,2,\dots,n$;
(C2) $C_n(x_1,x_2,\dots,x_n) = 0$ when $x_j = 0$ for at least one index $j \in \{1,2,\dots,n\}$;
(C3) the V_C-volume of every n-box $[\mathbf{a},\mathbf{b}]$ is non-negative $V_C([\mathbf{a},\mathbf{b}]) \geq 0$.

It follows from the above properties that a copula C satisfies the Lipschitz conditions, namely, for all points $\mathbf{u} = (u_1,u_2,\dots,u_n)$ and $\mathbf{u}' = (u_1',u_2',\dots,u_n')$ in $[0,1]^n$

$$|C(\mathbf{u}') - C(\mathbf{u})| \leq \sum_{j=1}^{n} |u_j' - u_j|. \tag{2.2}$$

Two of the most important n-copulas are the product copula

$$\Pi(u_1,u_2,\dots,u_n) := \prod_{j=1}^{n} u_j$$

and the greatest n-copula

$$M_n(u_1,u_2,\dots,u_n) := \min\{u_1,u_2,\dots,u_n\}.$$

For every $(u_1,u_2,\dots,u_n) \in [0,1]^n$ and for every n-copula C, one has

$$W_n(u_1,u_2,\dots,u_n) \leq C(u_1,u_2,\dots,u_n) \leq M_n(u_1,u_2,\dots,u_n),$$

where $W_n(u_1,u_2,\dots,u_n) := \max\{u_1+u_2+\cdots+u_n-n+1,0\}$; it is important to emphasize that W_n is an n-copula if, and only if, $n=2$, otherwise, i.e., for $n \geq 3$, it is a proper quasi-copula (see Ref. [6]). The set of n-copulas will be denoted by $\mathcal{C}_n$; if $n=2$, we shall simply write $\mathcal{C}$.

More on copulas can be read in Refs. [13, 11].

Definition 2.1. Let J be a finite or countable subset of the natural numbers $\mathbb{N}$, $\operatorname{card}(J) \leq \aleph_0$, let $(]a_k,b_k[)_{k\in J}$ be a family of sub-intervals of the unit interval $[0,1]$ indexed by J and let $(C_k)_{k\in J}$ be a family of copulas also indexed by J. It is required that any two of the intervals $]a_k,b_k[$ $(k \in J)$ have at most an endpoint in common. Then the *ordinal sum* C of $(C_k)_{k\in J}$ with respect to the family of intervals $(]a_k, b_k[)_{k\in J}$ is defined, for all $\mathbf{u} = (u_1, u_2, \dots, u_n) \in [0,1]^n$ by

$$C(\mathbf{u}) := \begin{cases} a_k + (b_k - a_k)\, C_k\left(\frac{\min\{u_1, b_k\} - a_k}{b_k - a_k}, \dots, \frac{\min\{u_n, b_k\} - a_k}{b_k - a_k}\right), \\ \qquad \text{if } \min\{u_1, u_2, \dots, u_n\} \in \,]a_k, b_k[\, \text{ for some } k \in J, \\ \min\{u_1, u_2, \dots, u_n\}, \qquad \text{elsewhere.} \end{cases} \tag{2.3}$$

For such a C one writes

$$C = (\langle a_k, b_k, C_k\rangle)_{k\in J}\,.$$

We wish to prove that the ordinal sum of n-copulas is again an n-copula for all $n \geq 2$. This result is part of the "shared knowledge" on 2-copulas, but a serious perusal of the literature shows that, although this result is quoted, for instance in Nelsen's monograph [11], it is impossible to trace its proof, at least to the best of the authors' knowledge; hence, we provide it below.

Theorem 2.1. *The ordinal sum* (2.3) *of the family of copulas* $(C_k)_{k\in J}$ *with respect to the family of intervals* $(]a_k, b_k[)_{k\in J}$ *is a copula.*

Proof. Since the boundary conditions (C1) and (C2) are easily verified, only the n-increasing property (C3) has to be established. This will be achieved in three steps.
Step 1: Consider first the ordinal sum

$$C := (\langle 0, a, C_1\rangle, \langle a, 1, C_2\rangle)\,,$$

where C_1 and C_2 are arbitrary n-copulas, possibly equal to M_n. Thus, $\mathrm{card}(J) = 2$. Any n-box $[\mathbf{u}, \mathbf{v}]$ in $[0,1]^n$ may be decomposed into a finite number of n-boxes $[\mathbf{u}_k, \mathbf{v}_k]$ with the following properties

- at most one of them, say $[\mathbf{u}_1, \mathbf{v}_1]$, is contained in $[0,a]^n$, and in this case $\mathbf{u}_1 = \mathbf{u}$;
- at most one of them, say $[\mathbf{u}_2, \mathbf{v}_2]$, is contained in $[a,1]^n$; and then $\mathbf{v}_2 = \mathbf{v}$;
- all the remaining n-boxes into which $[\mathbf{u}, \mathbf{v}]$ has been decomposed are intersected by the diagonal of the hypercube $[0,1]^n$ at one vertex.

When $[\mathbf{u}_1, \mathbf{v}_1]$ is contained in $[0,a]^n$, one has for the C-volume of $[\mathbf{u}_1, \mathbf{v}_1]$

$$V_C\left([\mathbf{u}_1, \mathbf{v}_1]\right) = a\, V_{C_1}\left(\left[\frac{\mathbf{u}_1}{a}, \frac{\mathbf{v}_1}{a}\right]\right),$$

which is non-negative, since C_1 is a copula.

In the same way, when $[\mathbf{u}_2, \mathbf{v}_2]$ is contained in $[a,1]^n$, one has for the C-volume of $[\mathbf{u}_2, \mathbf{v}_2]$

$$V_C\left([\mathbf{u}_2, \mathbf{v}_2]\right) = (1-a)\, V_{C_2}\left(\left[\frac{\mathbf{u}_2 - a}{1-a}, \frac{\mathbf{v}_2 - a}{1-a}\right]\right),$$

which is also non-negative, since C_2 is a copula.

The C-volume of any one of the remaining n-boxes of the decomposition of $[\mathbf{u}, \mathbf{v}]$ equals its M_n-volume, but this latter one is equal to zero. Since the C-volume is additive, one has

$$V_C([\mathbf{u}, \mathbf{v}]) \geq 0.$$

which shows that C is n-increasing, and, hence, an n-copula.

Step 2: The general finite case, $\aleph_0 > \operatorname{card}(J) > 2$ is dealt with by induction. Assume that one has shown that every ordinal sum of k summands

$$C^{(k)} = (\langle a_j, b_j, C_j \rangle_{j=1,2,\dots,k})$$

has been proved to be an n-copula and consider the ordinal sum of $k+1$ summands

$$C^{(k+1)} = (\langle a_j, b_j, C_j \rangle_{j=1,2,\dots,k+1}).$$

But then one has

$$C^{(k+1)} = \left(\langle a_1, b_k, C^{(k)} \rangle, \langle a_{k+1}, b_{k+1}, C_{k+1} \rangle \right),$$

which, as a consequence of the first part of the proof, is an n-copula.

Step 3: Let J be infinite, i.e. $\operatorname{card}(J) = \aleph_0$. For every $m \in \mathbb{N}$ denote by J_m the largest subfamily of J such that, for all indices k in J_m, the length of $]a_k, b_k[$ is at least $1/m$,

$$\forall\, k \in J_m \qquad b_k - a_k \geq \frac{1}{m}.$$

Obviously, J_m is finite, in fact, $\operatorname{card}(J_m) \leq m$; moreover, the sequence of sets $(J_m)_{m \in \mathbb{N}}$ is increasing, $J_m \subseteq J_{m+1}$ $(m \in \mathbb{N})$ and converges to J,

$$\lim_{m \to +\infty} J_m = \cup_{m \in \mathbb{N}} J_m = J.$$

For each $m \in \mathbb{N}$, let the copula $\widetilde{C}_m$ be the ordinal sum with respect to the family $(]a_k, b_k[)_{k \in J_m}$ constructed as in Steps 1 and 2 above. Thus, one obtains a decreasing sequence $(\widetilde{C}_m)_{m \in \mathbb{N}}$, which, as a consequence, has a pointwise limit; call this limit $\widetilde{C}$. For every rectangle $[\mathbf{a}, \mathbf{b}]$ contained in $[0,1]^n$, one has

$$V_{\widetilde{C}_m}([\mathbf{a}, \mathbf{b}]) \geq 0,$$

so that, taking the limit of the finite sum (2.1) with $C = \widetilde{C}_m$, one has $V_{\widetilde{C}}([\mathbf{a}, \mathbf{b}]) \geq 0$. Since the boundary conditions are obviously satisfied, this establishes the fact that $\widetilde{C}$ is a copula.

Now let $\mathbf{u} = (u_1, u_2, \dots, u_n)$ be a point in $[0,1]^n$. Assume that there exists $k \in J$ such that $\min\{u_1, u_2, \dots, u_n\}$ belongs to $]a_k, b_k[$ and let $m_0 = m_0(\mathbf{u})$ be the smallest natural number such that k belongs to J_{m_0}; then, by construction, $\widetilde{C}(\mathbf{u}) = \widetilde{C}_m(\mathbf{u})$ for every $m \geq m_0$. If, on the other hand, there is no index $k \in \mathbb{N}$ for which $\min\{u_1, u_2, \dots, u_n\}$ belongs to $]a_k, b_k[$, then one has $\widetilde{C}_m(\mathbf{u}) = \min\{u_1, u_2, \dots, u_n\}$ for every $m \in \mathbb{N}$ and, hence, also $\widetilde{C}(\mathbf{u}) =$

$\min\{u_1, u_2, \ldots, u_n\}$. This shows that $\widetilde{C}$ is the ordinal sum of the family of copulas $(C_k)_{k\in J}$ with respect to the family of intervals $(]a_k, b_k[)_{k\in J}$. □

3. Idempotents

Let C be an n-copula and let $\mathcal{I}_C$ be the set of idempotents of C, namely

$$\mathcal{I}_C := \{x \in [0,1] : C(x, x, \ldots, x) = x\}.$$

Notice that, for every copula C, the set $\mathcal{I}_C$ is not empty, since both 0 and 1 belong to it. The idempotents 0 and 1 will be called *trivial*.

Theorem 3.1. *The set $\mathcal{I}_C$ of idempotents of every copula C is closed. Moreover, if C is different from M_n, there exists a subset J of the set $\mathbb{N}$ of natural numbers, $J \subseteq \mathbb{N}$, and, for every $k \in J$, points a_k and b_k in $\mathcal{I}_C$ such that*

$$[0,1] \setminus \mathcal{I}_C = \cup_{k\in J}\,]a_k, b_k[\,. \tag{3.1}$$

Proof. That $\mathcal{I}_C$ is closed is an immediate consequence of the continuity of C. If $C = M_n$, then all the points of the unit interval are idempotent, viz. $\mathcal{I}_{M_n} = [0,1]$. Otherwise, the complement of $\mathcal{I}_C$ is not empty and, as the complement of a closed set in $\mathbb{R}$, it is open and, therefore, the union of an at most countable family of disjoint open intervals. □

Lemma 3.1. *Let C be an n-copula and $\mathcal{I}_C$ the set of its idempotents. If a is an idempotent for C, $a \in \mathcal{I}_C$, and $\min\{u_1, u_2, \ldots, u_n\} = a$, then*

$$C(u_1, u_2, \ldots, u_n) = a = \min\{u_1, u_2, \ldots, u_n\}.$$

Proof. Assume $u_j = a$; then

$$\begin{aligned} a = C(a, a, \ldots, a) &\le C(u_1, \ldots, u_{j-1}, a, u_{j+1}, \ldots, u_n) \\ &\le \min\{u_1, \ldots, u_{j-1}, a, u_{j+1}, \ldots, u_n\} = a, \end{aligned}$$

whence the assertion. □

This lemma ought to be compared to Proposition 2.3 (i) in [8].

Theorem 3.2. *With the same notation as in Theorem 3.1, if*

$$\mathcal{I}_C^n := \underbrace{\mathcal{I}_C \times \mathcal{I}_C \times \cdots \times \mathcal{I}_C}_{n \text{ times}},$$

then the support of a copula C lies entirely in the set $\mathcal{I}_C^n \bigcup \left(\cup_{k\in J} [a_k, b_k]^n\right)$.

Proof. The probability assigned by the copula C to the n-box $[a_k, b_k]^n$ is given by its V_C-volume

$$V_C\left([a_k, b_k]^n\right) = \sum_{\mathbf{c}} (-1)^{N(\mathbf{c})}\, C(\mathbf{c}),$$

where

- the sum is over the 2^n vertices of $[a_k, b_k]^n$;
- $\mathbf{c}$ is given by $\mathbf{c} = (c_1, c_2, \dots, c_n)$, with $c_j \in \{a_k, b_k\}$ for every $j = 1, 2, \dots, n$;
- $N(\mathbf{c})$ equals 1 or -1 according to whether the number of components of $\mathbf{c}$ that are equal to a_k is even, or odd, respectively.

Since a_k and b_k are idempotents of C, one has $C(b_k, b_k, \dots, b_k) = b_k$, while, by virtue of Lemma 3.1, $C(\mathbf{c}) = a_k$ for all the remaining vertices. Therefore,

$$V_C\left([a_k, b_k]^n\right) = b_k - a_k.$$

Consider the following alternative:

Case 1. If $[0,1] = \cup_{k\in J}\,[a_k, b_k]$, then the mass assigned by C to $[a_k, b_k]^n$ equals

$$\sum_{k\in J} V_C\left([a_k, b_k]^n\right) = \sum_{k\in J}(b_k - a_k) = 1.$$

In this case C concentrates all the probability on the set $\cup_{k\in J}\,[a_k, b_k]^n$.

Case 2. If $\cup_{k\in J}\,[a_k, b_k]$ is strictly included in the unit interval $[0,1]$, namely

$$[0,1] \setminus \cup_{k\in J}\,[a_k, b_k] \neq \emptyset,$$

then the probability C assigns to the segment of endpoints $(b_{k-1}, b_{k-1}, \dots, b_{k-1})$ and $(a_k, a_k, \dots, a_k)$ equals $a_k - b_{k-1}$ so that the total mass of $\mathcal{I}_C^n \bigcup \left(\cup_{k\in J}\,[a_k, b_k]^n\right)$ is

$$\sum_{k\in J} V_C\left([a_k, b_k]^n\right) + \sum_{k\in J}(a_k - b_{k-1}) = 1.$$

This concludes the proof. $\square$

Lemma 3.2. *Let β be in $]0,1]$ and let f be a non-decreasing and 1-Lipschitz function from $[0,\beta]$ into itself such that $f(0) = 0$ and $f(\beta) = \beta$. Then $f(t) = t$ for every $t \in [0,\beta]$.*

Proof. The 1-Lipschitz condition yields, for every $t \in]0,\beta]$,

$$f(t) = f(t) - f(0) \leq t$$

and

$$\beta - f(t) = f(\beta) - f(t) \leq \beta - t,$$

which together establish the assertion. $\square$

Our main result is contained in the following theorem

Theorem 3.3. *Let C be an n-copula different from M_n and let $\mathcal{I}_C$ be the set of its idempotents. Then, with the same notation as in Theorem 3.1,*

(a) *for every $k \in J$, the function $C_k : [0,1]^n \to [0,1]$ defined by*

$$C_k(\mathbf{u}) := \frac{C(a_k + \mathbf{u}\,(b_k - a_k)) - a_k}{b_k - a_k} \qquad \mathbf{u} = (u_1, u_2, \dots, u_n) \in [0,1]^n \tag{3.2}$$

is an n-copula;

(b) *C is the ordinal sum of $(C_k)_{k\in J}$ with respect to the family of intervals $(]a_k, b_k[)_{k\in J}$, viz.*

$$C = (\langle a_k, b_k, C_k\rangle)_{k\in J}\,.$$

Proof. (a) It is a straightforward consequence of the definition (3.2) that each C_k is n-increasing. Therefore, in order to prove that for every $k \in J$, C_k is an n-copula, it suffices to show that it satisfies the boundary conditions (C1) and (C2). We first prove that, for every $k \in J$, one has

$$C_k(1, 1, \ldots, 1) = 1 \qquad \text{and} \qquad C_k(0, 0, \ldots, 0) = 0.$$

Recalling that a_k and b_k are both idempotents of C one has

$$C_k(1, 1, \ldots, 1) = \frac{C(b_k, b_k, \ldots, b_k) - a_k}{b_k - a_k} = \frac{b_k - a_k}{b_k - a_k} = 1 \tag{3.3}$$

and

$$C_k(0, 0, \ldots, 0) = \frac{C(a_k, a_k, \ldots, a_k) - a_k}{b_k - a_k} = \frac{a_k - a_k}{b_k - a_k} = 0.$$

Now set $u_1 = \cdots = u_{j-1} = u_{j+1} = \cdots = u_n = 1$ in (3.2) in order to obtain

$$\begin{aligned} C_k(1, \ldots, 1, t, 1, \ldots, 1) \;&=\; \frac{C(b_k, \ldots, b_k, a_k + t\,(b_k - a_k), b_k, \ldots, b_k) - a_k}{b_k - a_k} \\ &=: \varphi_k(t). \end{aligned}$$

It follows form (3.3) that $\varphi_k(1) = 1$, while, because of Lemma 3.1,

$$\varphi_k(0) = \frac{C(b_k, \ldots, b_k, a_k, b_k, \ldots, b_k) - a_k}{b_k - a_k} = \frac{a_k - a_k}{b_k - a_k} = 0.$$

Thus, Lemma 3.2 applied to φ_k, with $\beta = 1$, yields $\varphi_k(t) = t$ for every $t \in [0, 1]$; as a consequence, C_k satisfies the boundary condition (C1).

Next, set $u_j = 0$, in order to obtain, with the position $s_i = a_k + u_i\,(b_k - a_k)$ ($j = 1, \ldots, j-1, j+1, \ldots, n$),

$$\begin{aligned} C_k(u_1, \ldots, u_{j-1}, 0, u_{j+1}, \ldots, u_n) &= \frac{C(s_1, \ldots, s_{j-1}, a_k, s_{j+1}, \ldots, s_n) - a_k}{b_k - a_k} \\ &= \frac{a_k - a_k}{b_k - a_k} = 0, \end{aligned}$$

by virtue of Lemma 3.1. Therefore, each C_k satisfies also the boundary condition (C2).

(b) In order to show that C is the ordinal sum of the C_k's, the following cases will have to be considered.

Case 1. There is an index $k \in J$ such that all the components of $(u_1, u_2, \ldots, u_n)$ belong to $]a_k, b_k[$. Then (3.2) yields

$$C(u_1, u_2, \ldots, u_n) = a_k + (b_k - a_k)\, C_k\left(\frac{u_1 - a_k}{b_k - a_k}, \frac{u_2 - a_k}{b_k - a_k}, \ldots, \frac{u_n - a_k}{b_k - a_k}\right).$$

Case 2. $\min\{u_1, u_2, \dots, u_n\}$ belongs to $\mathcal{I}_C$; then, on account of Lemma 3.1,

$$C(u_1, u_2, \dots, u_n) = \min\{u_1, u_2, \dots, u_n\}.$$

Case 3. The point $\mathbf{u} = (u_1, u_2, \dots, u_n)$ does not belong to $\mathcal{I}_C^n \bigcup (\cup_{k\in J} [a_k, b_k]^n)$. Set $u_j = \min\{u_1, u_2, \dots, u_n\}$ and assume that this minimum belongs to the interval $]a_i, b_i[$. The value $C(\mathbf{u})$ equals the probability of the n-box $\prod_{j=1}^n [0, u_j]$. In its turn, this probability is given, on account of Theorem 3.2, by

$$a_i + V_C \left(\prod_{\substack{j=1 \\ j\neq i}}^{n} [a_j, b_j] \times [a_i, u_i] \right).$$

Now, because of Lemma 3.1, $C(x_1, \dots, x_{i-1}, a_i, x_{i+1}, \dots, x_n) = a_i$ for $x_j \in \{a_j, b_j\}$ $(j \neq i)$; therefore

$$\begin{aligned} C(u_1, u_2, \dots, u_n) &= a_i + C(b_i, \dots, b_i, u_i, b_i, \dots, b_i) - a_i \\ &= a_i + (b_i - a_i)\, C_i \left(1, \dots, 1, \frac{u_i - a_i}{b_i - a_i}, 1, \dots, 1\right) \\ &= u_i = \min\{u_1, u_2, \dots, u_n\}. \end{aligned}$$

This shows that C is indeed the ordinal sum of the C_k's and concludes the proof. $\square$

As a consequence of the previous theorem, one can prove the following corollaries.

Corollary 3.1. *Any subinterval* $[a, b]$ *of the unit interval* $[0, 1]$ *is the set of idempotents of a copula* C*, namely there exists* $C \in \mathcal{C}_n$ *such that* $\mathcal{I}_c = [a, b]$.

Proof. The case $a = 0$ and $b = 1$ is trivial, because only for the copula M_n does one have $\mathcal{I}_{M_n} = [0, 1]$.

If $0 < a < b < 1$, then it suffices to consider, for instance, the ordinal sum

$$(\langle 0, a, \Pi_n\rangle, \langle b, 1, \Pi_n\rangle).$$

On the other hand, if $a = 0$, then the ordinal sum

$$(\langle 0, b, M_n\rangle \langle b, 1, \Pi_n\rangle)$$

has the interval $[a, b]$ as its set of idempotents. One proceeds in a similar manner in the case $a > 0$, $b = 1$. $\square$

Corollary 3.2. *A copula* C *with only trivial idempotents is ordinally irreducible, viz. it admits only the representation* $(\langle 0, 1, C\rangle)$.

Corollary 3.3. *Every copula* C *different from* M_n *can be written as the ordinal sum of copulas having only trivial idempotents.*

Remark 3.1. The ordinal sum of copulas discussed till now is related to the main diagonal section of the domain $[0,1]^n$ (the idempotents of a copula C are exactly the fixed points of its diagonal section) and it is based on the strongest copula M_n. For $n = 2$, similar considerations were presented for the opposite diagonal section (namely, their null points were of interest) and based on the weakest 2-copula W, the so called W-ordinal sum introduced in Ref. [10] (see also Ref. [5]). Later it was shown that all results concerning W-ordinal sums of 2-copulas can be obtained from the standard ordinal sums of 2-copulas, exploiting the fact that for a given copula $C : [0,1]^2 \to [0,1]$, also the functions $C_1, C_2 : [0,1]^2 \to [0,1]$ defined by $C_1(x,y) = x - C(x, 1-y)$ and $C_2(x,y) = y - C(1-x, y)$ are 2-copulas, and if $C(a,a) = a$ (i.e., a is an idempotent element of C, and, hence, a fixed point its diagonal section, then $C_1(a, 1-a) = C_2(1-a, a) = 0$ (i.e., the point a is the null point of C_1-opposite diagonal section, $1-a$ is the null point of C_2-opposite diagonal section). Similar considerations may be presented for any dimension $n > 2$; only, then one should deal with a diagonal of hypercube $[0,1]^n$ connecting some neighbouring vertex of the top vertex $(1,1,\ldots,1)$ with its dual vertex. For instance, for $n = 3$, one can choose the vertices $(1,1,0)$ and $(0,0,1)$, i.e., deal with the diagonal $\{(1-a, 1-a, a) \mid a \in [0,1]\}$. In such a case, since for any 3-copula C, also the function $C^* : [0,1]^3 \to [0,1]$ given by $C^*(x,y,z) = z - C(1-x, 1-y, z)$ is a 3-copula, one can discuss the structure of 3-copulas related to the null points of the diagonal section $\delta : [0,1] \to [0,1]$ given by $\delta(z) = C^*(1-z, 1-z, z) = z - C(z,z,z) = z - \delta(z)$, where $\delta : [0,1] \to [0,1]$ is the main diagonal section of C.

4. Associative copulas

Each associative 2-copula C is a continuous t-norm [13,8] and, because of the representation theorem for continuous t-norms, it is an ordinal sum of Archimedean copulas [13,8,11]. As a consequence of the associativity of C, there is a genuine extension $C_{(n)} : [0,1]^n \to [0,1]$ of C to an n-ary function ($n \in \mathbb{N}$), $n > 2$ inductively defined by

$$C_{(2)}(u_1, u_2) := C(u_1, u_2),$$
$$C_{(n)}(u_1, u_2, \ldots, u_n) := C\left(C_{(n-1)}(u_1, u_2, \ldots, u_{n-1}), u_n\right) \qquad (n \geq 3).$$

By induction, it is not difficult to check that the functions C and $C_{(n)}$ have all the same set of idempotents, and that by Theorem 3.3, $C_{(n)}$ is an n-ary copula if, and only if, all n-ary extensions of the Archimedean 2-copulas related to C are n-copulas.

For Archimedean 2-copulas, i.e., copulas $C : [0,1]^2 \to [0,1]$ generated by a continuous convex strictly decreasing function $f : [0,1] \to [0,\infty]$, $f(1) = 0$, such that

$$C(x,y) = f^{-1}(\min\{f(0), f(x) + f(y)\}) \tag{4.1}$$

(note that f is then called a *generator* of C), the corresponding n-ary extension $C_{(n)} : [0,1]^n \to [0,1]$ is given by

$$C(x_1, x_2, \dots, x_n) = f^{-1}(\min\{f(0), f(x_1) + f(x_2) + \cdots + f(x_n)\}). \tag{4.2}$$

We owe to Kimberling [7] the following important result.

Theorem 4.1. *For a given Archimedean copula $C : [0,1]^2 \to [0,1]$ generated by a function $f : [0,1] \to [0,\infty]$ the following statements are equivalent:*

1. *for each $n \in \mathbb{N}$, $n > 2$, the function $C_{(n)} : [0,1]^n \to [0,1]$ given by (4.2) is an n-copula;*
2. *the pseudo-inverse $f^{(-1)} : [0,\infty] \to [0,1]$ of f defined by*

$$f^{(-1)}(u) = f^{-1}(\min\{f(0), u\})$$

is a completely monotone function, i.e., it has derivatives of every order on $]0,\infty[$ and these derivatives alternate their sign (i.e., each odd derivative is negative and each even derivative is positive).

Note that, on account of (2.), each one of the copulas characterized in Theorem 4.1 is strict and thus the pseudo-inverse of f coincides with its standard inverse, $f^{(-1)} = f^{-1}$.

The above result allows us to characterize the class of all associative copulas with the property that their n-ary forms are n-copulas for arbitrary n.

Corollary 4.1. *A function $C : [0,1]^2 \to [0,1]$ is an associative copula such that for all n in $\mathbb{N}$ with $n > 2$, the n-ary extension $C_{(n)} : [0,1]^n \to [0,1]$ of C is an n-copula, if, and only, if there is a disjoint system $(]a_k, b_k[)_{k\in J}$ of open subintervals of $[0,1]$ and a system $(g_k)_{k\in J}$ of completely monotone bijections $g_k :]0,\infty] \to [a_k, b_k[$, such that*

$$C(x,y) = \begin{cases} g_k^{-1}\left(g_k(x) + g_k(y)\right), & \textit{if } (x,y) \in\,]a_k, b_k[^2 \textit{ for some } k \in J, \\ \min\{x,y\}, & \textit{elsewhere.} \end{cases}$$

Remark 4.1. Recently, Radojević ([12]) has introduced a pseudo-product

$$\otimes : \cup_{n\in\mathbb{N}} [0,1]^n \to [0,1],$$

which is a background for a generalization of Boolean forms. This pseudo-product is, in fact, an associative extended aggregation function each one of whose restrictions to $[0,1]^n$, $(n > 1)$, is an n-ary copula. Because of the required associativity of $\otimes$, Corollary 4.1 provides a complete characterization of all such pseudo-products.

Recently McNeil and Nešlehová [9] have characterized, for a fixed $n \geq 2$, the lower bounds B_n for Archimedean 2-copulas with the property that their n-ary forms are n-copulas as the Archimedean copulas B_n with additive generator $f_n : [0,1] \to [0,\infty]$ given by

$$f_n(x) = 1 - x^{1/(n-1)}.$$

Since $\lim_{n\to\infty} B_n = \Pi$, and since Π itself is an Archimedean copula such that its n-ary form is an n-copula for all $n \in \mathbb{N}$, with $n \geq 2$, we can conclude that Π is the smallest associative copula, for which all the n-ary forms are n-copulas. Therefore, all these copulas are necessarily positive quadrant dependent. This latter statement may be regarded as an extension of Corollary 4.6.3 in Ref. [11].

5. Appendix

We give a direct and explicit proof of Theorem 2.1 in the case $n = 2$. This proof avoids the recourse to the additivity of C-volumes, in order to include the case of a countably infinite number of summands. A sketch of the proof when the summands are of a finite number may be found in Ref. [1, Theorem 2.4.2 (c)].

Proof of Theorem 2.1. As was said above we shall limit ourselves to the case $n = 2$; then the ordinal sum in question is defined by

$$C(u,v) := \begin{cases} a_k + (b_k - a_k)\, C_k\left(\frac{u-a_k}{b_k-a_k}, \frac{v-a_k}{b_k-a_k}\right), & (u,v) \in [a_k, b_k]^n, \\ \min\{u,v\}, & \text{elsewhere.} \end{cases}$$

It is clear that C satisfies the boundary conditions for a copula; therefore, only the 2-increasing property

$$\Delta C_{(u_1,u_2)}^{(v_1,v_2)} := C(u_2,v_2) + C(u_1,v_1) - C(u_2,v_1) - C(u_1,v_2) \geq 0, \qquad (5.1)$$

for every rectangle $[u_1,u_2] \times [v_1,v_2]$ in the unit square $[0,1]^2$, will have to be proved. Several cases will have to be considered.

There is nothing to prove if the rectangle $[u_1,u_2] \times [v_1,v_2]$ is entirely contained either in one of the squares $[a_k,b_k]^2$ $(k \in J)$ or in $[0,1]^2 \setminus \cup_{k\in J} [a_k,b_k]^2$.

Consider next the case of a rectangle with two vertices inside one of the squares $[a_k,b_k]^2$ and the remaining two ones outside, for instance, let the vertices (u_1,v_1) and (u_2,v_1) lie inside $[a_k,b_k]^2$ while (u_2,v_2) and (u_1,v_2) are placed above the square $[a_k,b_k]^2$. In this case, one has, because of the Lipschitz inequality (2.2),

$$\begin{aligned}\Delta C_{(u_1,u_2)}^{(v_1,v_2)} &= \min\{u_2,v_2\} - \min\{u_1,v_2\} - a - (b_k - a_k)\, C_k\left(\frac{u_2-a_k}{b_k-a_k}, \frac{v_1-a_k}{b_k-a_k}\right) \\ &\quad + a + (b_k - a_k)\, C_k\left(\frac{u_1-a_k}{b_k-a_k}, \frac{v_1-a_k}{b_k-a_k}\right)\end{aligned}$$

$$
\begin{aligned}
&= u_2 - u_1 - (b_k - a_k) \left\{ C_k \left(\frac{u_2 - a_k}{b_k - a_k}, \frac{v_1 - a_k}{b_k - a_k} \right) \right. \\
&\quad \left. - C_k \left(\frac{u_1 - a_k}{b_k - a_k}, \frac{v_1 - a_k}{b_k - a_k} \right) \right\} \\
&\geq u_2 - u_1 - (b_k - a_k) \left(\frac{u_2 - a_k}{b_k - a_k} - \frac{u_1 - a_k}{b_k - a_k} \right) = 0,
\end{aligned}
$$

so that inequality (5.1) is satisfied.

All the other cases in which two vertices belong to a square $[a_k, b_k]^2$ and two to $[0,1]^2 \setminus \cup_{k \in J} [a_k, b_k]^2$ are dealt with in a similar manner.

Assume now that three vertices lie in $[0,1]^2 \setminus \cup_{k \in J} [a_k, b_k]^2$ and one belongs to $[a_k, b_k]^2$. One should now consider four possible cases; since they are similar, we shall treat only one of them, namely when only the upper right vertex (u_2, v_2) belongs to the square $[a_k, b_k]^2$. In this case

$$
\begin{aligned}
\Delta C_{(u_1,u_2)}^{(v_1,v_2)} &= a_k + (b_k - a_k)\, C_k \left(\frac{u_2 - a_k}{b_k - a_k}, \frac{v_2 - a_k}{b_k - a_k} \right) + v_1 - v_1 - u_1 \\
&\geq a_k + (b_k - a_k)\, W \left(\frac{u_2 - a_k}{b_k - a_k}, \frac{v_2 - a_k}{b_k - a_k} \right) - u_1 \\
&= a_k + \max\{u_2 - a_k + v_2 - b_k, 0\} - u_1.
\end{aligned}
$$

If $\delta := u_2 - a_k + v_2 - b_k > 0$, then

$$
\Delta C_{(u_1,u_2)}^{(v_1,v_2)} \geq u_2 + v_2 - b_k - u_1 > a_k - u_1 > 0.
$$

Instead if $\delta \leq 0$, then

$$
\Delta C_{(u_1,u_2)}^{(v_1,v_2)} \geq a_k - u_1 > 0.
$$

Thus, also in this case inequality (5.1) is satisfied.

Finally, consider the situation in which the right upper vertex is in the square $[a_k, b_k]^2$ and the left lower vertex lies in the square $[a_j, b_j]$ with $j < k$. Then

$$
\begin{aligned}
\Delta C_{(u_1,u_2)}^{(v_1,v_2)} &= a_k + (b_k - a_k)\, C_k \left(\frac{u_2 - a_k}{b_k - a_k}, \frac{v_2 - a_k}{b_k - a_k} \right) \\
&\quad + a_j + (b_j - a_j)\, C_j \left(\frac{u_2 - a_j}{b_j - a_j}, \frac{v_2 - a_j}{b_j - a_j} \right) - u_1 - v_1 \\
&\geq a_k + \max\{u_2 - a_k + v_2 - a_k - b_k + a_k, 0\} \\
&\quad + a_j + \max\{u_1 - a_j + v_1 - a_j - b_j + a_j, 0\} - u_1 - v_1.
\end{aligned}
$$

Setting $\delta_2 := u_2 - a_k + v_2 - b_k$ and $\delta_1 := u_1 - a_j + v_1 - b_j$, one has four cases to consider.

Case 1. If $\delta_2 > 0$ and $\delta_1 > 0$, then

$$\begin{aligned}\Delta C_{(u_1,u_2)}^{(v_1,v_2)} &\geq u_2 + v_2 - b_k + u_1 + v_1 - b_j - u_1 - v_1 = (u_2 + v_2) - (b_j + b_k)\\ &\geq (u_2 + v_2) - (a_k + b_k) > 0,\end{aligned}$$

since $b_j \leq a_k$ and $\delta_2 > 0$.

Case 2. If $\delta_2 > 0$ and $\delta_1 \leq 0$, then

$$\begin{aligned}\Delta C_{(u_1,u_2)}^{(v_1,v_2)} &\geq u_2 + v_2 - b_k + a_j - u_1 - v_1 \geq (u_2 + v_2) - (b_j + b_k)\\ &\geq u_2 + v_2 - (a_k + b_k) > 0,\end{aligned}$$

since $b_j \leq a_k$.

Case 3. If $\delta_2 \leq 0$ and $\delta_1 > 0$, then

$$\Delta C_{(u_1,u_2)}^{(v_1,v_2)} = a_k - b_j \geq 0.$$

Case 4. If $\delta_2 \leq 0$ and $\delta_1 \leq 0$, then

$$\Delta C_{(u_1,u_2)}^{(v_1,v_2)} = a_k + a_j - u_1 - v_1 \geq 0,$$

since $\delta_1 \leq 0$ implies $u_1 + v_1 \leq a_j + b_j \leq a_j + a_k$.

Therefore, in every case one has $\Delta C_{(u_1,u_2)}^{(v_1,v_2)} \geq 0$, so that (5.1) is satisfied. This concludes the proof. □

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